Exercise 10

Prove that

(a) z is real if and only if $\bar{z} = z$;

(b) z is either real or pure imaginary if and only if $\bar{z}^2 = z^2$.

Solution

Part (a)

Suppose that z is real. Then $z = x + i0 = x$ and $\bar{z} = x - i0 = x$. Thus, $\bar{z} = z$.

Suppose that $\bar{z} = z$. Then $x - iy = x + iy$, or $-iy = iy$. This equation is only satisfied if $y = 0$. The imaginary component is zero, so z is real.

Therefore, z is real if and only if $\bar{z} = z$.

Part (b)

Suppose that z is real. Then $\bar{z} = z$ from part (a). Square both sides to get $\bar{z}^2 = z^2$.

Suppose that z is purely imaginary. Then $z = 0 + iy = iy$ and $\bar{z} = 0 - iy = -iy$. Then $z^2 = -y^2 = \bar{z}^2$.

Suppose that $\bar{z}^2 = z^2$. Then

$$\begin{aligned}(x - iy)^2 &= (x + iy)^2 \\ x^2 - 2ixy - y^2 &= x^2 + 2ixy - y^2 \\ -2ixy &= 2ixy \\ xy &= 0 \\ x = 0 \quad &\text{or} \quad y = 0.\end{aligned}$$

Thus, $z = x + iy$ is either real or purely imaginary.

Therefore, z is either real or purely imaginary if and only if $\bar{z}^2 = z^2$.